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A Class of Stochastic Evolutions That Scale to the Porous Medium Equation

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A class of reversible Markov jump processes on a periodic lattice is described and a result about their scaling behavior stated: Under diffusion scaling, the empirical measure converges to a solution of the porous medium equation on the d-dimensional torus. The process can be viewed as a randomly interacting configuration of sticks that evolves through exchanges of stick pieces between nearest neighbors through a zero-range pressure mechanism, with conservation of total stick length.

KEY WORDS: Porous medium equation; hydrodynamic scaling limit.

It is physically important to find the microscopic origins of equations describing macroscopic systems. A well-known problem of this kind is the derivation of the Euler and Navier–Stokes equations for fluids from the Hamiltonian dynamics of molecules. The objective of our study is to find a microscopic model of interacting random processes for a macroscopic system governed by the porous medium equation. In this brief note we only present the result; full details[2] will be published elsewhere.

The porous medium equation or PME in d space dimensions is the nonlinear heat equation

$$\partial_t u = \kappa \, \Delta(u^\alpha) \tag{1}$$

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where $u = u(t, \xi)$ is a scalar function of $(t, \xi) \in (0, \infty) \times \mathscr{R}^d$, Δ is the Laplacian acting on the ξ variable, and $\kappa > 0$ and $\alpha > 1$ are constants. The flow of an ideal gas through a porous medium is the application that named the equation. The derivation of the PME in this case involves the conservation of mass, Darcy's law, and equations of state. The thermodynamic nature of the flow determines the value of α. In detail, if m is the ratio between the specific heat at constant volume and the specific heat at constant pressure, then $\alpha = 1 + 1/m$. The value of m is 1 for an isothermal flow and 0.71 for air in an adiabatic flow. More details can be found in ref. 4. Writing $\Delta(u^\alpha) = \nabla \cdot (\alpha u^{\alpha - 1} \nabla u)$ shows that the PME is a valid model when diffusivity depends on concentration in a power-law fashion. Also, the degeneracy of the coefficient causes the disturbances to propagate with finite speed, which is invalid in the case of the linear heat equation. A recent mathematical survey of the PME and numerous references can be found in ref. 6.

Informally speaking, the stochastic process we study describes a configuration of sticks that evolves through random stick-breaking events. The rate at which a particular stick breaks depends on its length. The broken-off piece is added on to another randomly chosen stick, so that total stick length is conserved at all times. To be precise, fix $\alpha > 1$ in the PME and a large integer N (this will eventually be the scaling parameter). We arrange the sticks on the sites $i = (i_1, \ldots, i_d)$ of the finite d-dimensional lattice

$$\mathscr{L}_N^d = \{ i \in \mathscr{L}^d : 0 \leqslant i_k < N, k = 1, \ldots, d \}$$

with periodic boundary conditions. The state space of the process is $\mathscr{S}_N = [0, \infty)^{\mathscr{L}_N^d}$. An element of $\mathscr{S}_N$ is a configuration $x = (x_i : i \in \mathscr{L}_N^d)$, where x_i denotes the length of the stick at site i. The state evolves on the space $\mathscr{S}_N$ as a Markov jump process according to the following rule: If the current state is x, then each stick x_i waits an exponentially distributed random time with expectation $(\alpha - 1) x_i^{1-\alpha}$ to break. If stick x_k is the first one to break, pick a random length u distributed on the interval $[0, x_k]$ with density $(\alpha - 1) x_i^{1-\alpha} u^{\alpha - 2}$, and a neighbor l of k at random on the lattice (observing periodic boundary conditions), then transfer the piece u from site k onto site l. The new state becomes

$$x_i^{u,k,l} = \begin{cases} x_i, & i \neq k, l \\ x_k - u, & i = k \\ x_l + u, & i = l \end{cases}$$

After the jump to $x^{u,k,l}$, all sticks resume waiting.

The physical reason for the wild microscopic motions of the gas particles becoming organized in the macroscopic scale is the local conservation laws of mass, momentum, and energy. In our model one may view each stick length as a measure of the mass of a certain material. The conservation of total stick length corresponds to the law of conservation of mass for the gas. The key feature of this microscopic dynamics is that the jump rate depends on the broken-off piece in a power-law fashion, in analogy with the macroscopic model.

Let Φ_ρ^N, $\rho \geqslant 0$, denote the probability measure on $\mathscr{S}_N$ under which the x_i are independent exponentially distributed random variables with common expectation ρ. Here $\{\Phi_\rho^N : \rho \geqslant 0\}$ is a one-parameter family of reversible invariant measures for the process; that is, each Φ_ρ^N is invariant under the dynamics and detailed balance holds.

The dynamics on $\mathscr{S}_N$ can also be described in terms of the evolution of the probability density of the process: If f_t^N denotes the probability density of the process at time $t \geqslant 0$ relative to $\Phi^N := \Phi_1^N$, then f_t^N solves the forward (Fokker–Planck) equation

$$\frac{\partial}{\partial t} f_t^N = L_N f_t^N$$

where the operator L_N is defined by

$$L_N f(x) := \sum_{k \in \mathscr{T}_N^d} \frac{1}{2d} \sum_{\substack{l \in \mathscr{T}_N^d \\ |l-k|=1}} \int_0^{x_k} u^{\alpha-2} [f(x^{u,k,l}) - f(x)]\, du, \qquad x \in \mathscr{S}_N \tag{2}$$

Pick and fix now an initial density f_0^N for the process on $\mathscr{S}_N$ for each positive integer N. Denote by

$$x^N(t) = (x_i^N(t) : i \in \mathscr{L}_N^d), \qquad t \geqslant 0$$

the random evolution of the configuration. We follow the profile of the sticks by studying the empirical measure

$$\mu_t^N(d\theta) = \frac{1}{N^d} \sum_{i \in \mathscr{T}_N^d} x_i^N(t)\, \delta_{i/N}(d\theta), \qquad \theta \in \mathscr{T}^d$$

Here $\mathscr{T}^d = \mathscr{R}^d/\mathscr{L}^d$ is the d-dimensional standard torus, and μ_t^N is regarded as a random measure on $\mathscr{T}^d$.

The basic assumption is that a law of large numbers holds at time 0: As $N \to \infty$, μ_0^N converges in probability to some finite measure ν on $\mathscr{T}^d$ as $N \to \infty$. This is an assumption on the sequence $\{f_0^N\}_{N=1}^\infty$ of initial

densities. More precisely, if $P_0^N(dx) = f_0^N(x)\, \Phi^N(dx)$ denotes the initial distribution of the process, then for all smooth test functions J on $\mathscr{T}^d$ and every $\delta > 0$,

$$\lim_{N \to \infty} P_0^N \left\{ x \in \mathscr{S}_N : \left| \frac{1}{N^d} \sum_{i \in \mathscr{L}_N^d} J\left(\frac{i}{N}\right) x_i - \int_{\mathscr{T}^d} J(\theta)\, v(d\theta) \right| \geqslant \delta \right\} = 0 \qquad (3)$$

Notice that the spatial scaling was already incorporated in the definition of μ_t^N. The temporal scaling is introduced by explicitly multiplying time by the factor N^2. Under (3) and (4), (5) below, we have:

Theorem 1. For every $t > 0$, the empirical measure $\mu_{N^2 t}^N(d\theta)$ converges in probability to a deterministic measure $u(t, \theta)\, d\theta$ as $N \to \infty$. The limiting profile $u(t, \theta)$ is characterized as the weak solution of the PME (1) on the torus $\mathscr{T}^d$, with $\kappa = (\alpha - 1)!/(2d)$ and initial data $u(0, \theta)\, d\theta = v(d\theta)$.

The last statement means that for all smooth test functions J on $\mathscr{T}^d$,

$$\int_{\mathscr{T}^d} J(\theta)\, u(t, \theta)\, d\theta - \int_{\mathscr{T}^d} J(\theta)\, v(d\theta) = (\alpha - 1)! \int_0^t ds \int_{\mathscr{T}^d} u^\alpha(s, \theta)\, \Delta J(\theta)\, d\theta$$

and uniqueness is in the Lebesgue space $L^{\alpha+1}((0, \infty) \times \mathscr{T}^d)$.

In addition to the basic assumption (3), we need two technical assumptions at time zero. In dimensions $d \geqslant 2$, an initial moment bound is needed for *a priori* moment bounds for all $t > 0$:

$$\sup_N N^{-d} \int_{\mathscr{S}_N} \sum_{i \in \mathscr{L}_N^d} x_i^2 f_0^N(x)\, \Phi^N(dx) < +\infty \qquad (4)$$

Second, to prove local equilibrium we utilize a refinement[5] of the entropy method of Guo *et al.*[3] and for this we need an entropy bound at time zero, in all dimensions:

$$H(f_0^N) := \int_{\mathscr{S}_N} f_0^N \log f_0^N\, d\Phi^N = o(N^{d+2}) \qquad \text{as} \quad N \to \infty \qquad (5)$$

Some concluding remarks: As far as we know, the class of processes with generators (2) has not been introduced before, except for the special case $\alpha = 2$ studied by Suzuki and Uchiyama[5] and Ekhaus and Seppäläinen.[1] It would be desirable to relax assumptions (4) and (5), especially the latter, to permit the study of the microscopic nature of the free boundary of the PME. Note that (5) prevents empty sites in the initial stick configuration. (This drawback can be lifted for $\alpha = 2$; see ref. 1) However, we do not require the macroscopic initial profile v to charge all

of $\mathcal{T}^d$. For example, we can take $\nu(d\theta) = u_0(\theta)\, d\theta$ for any bounded, continuous function $u_0(\theta) \geqslant 0$. In the case $d = 1$, ν can be any nonatomic finite measure or even a singular point mass $\nu = \delta_0$. In all these cases we can take $P_0^N(dx)$ to be an appropriate independent product with entropy $H(f_0^N) = \mathcal{O}(N^d \log N)$.

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